Chapter 11  Vectors and the Geometry of Space

Chapter Summary

Section Topics

11.1  **Vectors in the Plane**—Write the component form of a vector. Perform vector operations and interpret the results geometrically. Write a vector as a linear combination of standard unit vectors. Use vectors to solve problems involving force or velocity.

11.2  **Space Coordinates and Vectors in Space**—Understand the three-dimensional rectangular coordinate system. Analyze vectors in space. Use three-dimensional vectors to solve real-life problems.

11.3  **The Dot Product of Two Vectors**—Use properties of the dot product of two vectors. Find the angle between two vectors using the dot product. Find the direction cosines of a vector in space. Find the projection of a vector onto another vector. Use vectors to find the work done by a constant force.

11.4  **The Cross Product of Two Vectors in Space**—Find the cross product of two vectors in space. Use the triple scalar product of three vectors in space.

11.5  **Lines and Planes in Space**—Write a set of parametric equations for a line in space. Write a linear equation to represent a plane in space. Sketch the plane given by a linear equation. Find the distances between points, planes, and lines in space.

11.6  **Surfaces in Space**—Recognize and write equations of cylindrical surfaces. Recognize and write equations of quadric surfaces. Recognize and write equations of surfaces of revolution.

11.7  **Cylindrical and Spherical Coordinates**—Use cylindrical coordinates to represent surfaces in space. Use spherical coordinates to represent surfaces in space.

Chapter Comments

All of the ideas in this chapter need to be discussed in order for your students to have a good understanding of vectors. Point out to your students that vectors are not little pointed arrows, but that a directed line segment is just our way of representing a vector geometrically. Also note that this geometric representation of a vector does not have location. It does have direction and magnitude. On page 785 in the text is a Note about the words perpendicular, orthogonal, and normal. This is worth discussing with your students because sometimes it seems that these words are used interchangeably.

The dot product of two vectors is sometimes referred to as scalar multiplication and the cross product as vector multiplication. The reason for this is because the dot product is a *scalar* and the cross product is a *vector*. Point out this distinction to your students. The way to find a cross product is to use a 3 by 3 determinant. You will probably have to show your students how to calculate this.

When discussing lines and planes in space, Section 11.5, point out to your students that direction numbers are not unique. If \( a, b, c \) is a set of direction numbers for a line or a plane, then \( ka, kb, kc \) where \( k \in \mathbb{R}, k \neq 0 \), is also a set of direction numbers for that line or plane.

The distance formulas in Section 11.5 between a point and a plane and between a point and a line need not be memorized. However, go over these so that the students know where they are when they need to look them up.
It is important for your students to be familiar with the surfaces in space discussed in Section 11.6. The concepts in Chapters 14 and 15 will be much easier if the student immediately recognizes from the given equation which of the six surfaces on pages 814 and 815 is under discussion.

Two alternate coordinate systems are discussed in Section 11.7. The cylindrical coordinate system should be familiar to your students because it is an extension of the polar coordinate system in the plane. However, the spherical coordinate system will probably be new to your students. Time spent here understanding this system will be rewarded in Chapters 14 and 15 when you are performing multiple integrations.

**Section 11.1  Vectors in the Plane**

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**Exercises 9–16 (9–16 in Calculus 8/e)**
We rewrote the direction line to include a new part (c) that asks students to write the vector as the linear combination of the standard unit vectors \( \textbf{i} \) and \( \textbf{j} \) (see Example 5).

**Exercises 31–36 (31–36 in Calculus 8/e)**
We reordered the exercises to improve the grading.

**Exercises 47–50 (47–50 in Calculus 8/e)**
We reordered the exercises to improve the grading.

**Exercises 69–74 (69–74 in Calculus 8/e)**
We rewrote the direction line, replacing “normal” with “perpendicular,” because we have not defined normal.

**Capstone**

**Page 773, Exercise 78** You can use this exercise to review the following concepts.

- Writing the component form of a vector
- Writing a vector as a linear combination of standard unit vectors
- Sketching a vector
- Finding the magnitude of a vector

Go over the above concepts and the solution below.

**Solution**

(a) \( \mathbf{v} = \langle 9 - 3, 1 - (-4) \rangle = \langle 6, 5 \rangle \)

(b) \( \mathbf{v} = 6\mathbf{i} + 5\mathbf{j} \)

(c)

(d) \( \| \mathbf{v} \| = \sqrt{6^2 + 5^2} = \sqrt{61} \)
Section 11.2   Space Coordinates and Vectors in Space

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Exercises 1–6 (1–6 in Calculus 8/e)
We reordered the exercises to improve the grading.

Exercises 49–52 (49–52 in Calculus 8/e)
We rewrote the direction line to include a new part (b) asking students to write the vector using standard unit vector notation (see Example 6).

Exercises 57 and 58 (57 and 58 in Calculus 8/e)
We rewrote the direction line to include a new part (c) asking students to write the vector using standard unit vector notation (see Example 6).

Capstone

Page 781, Exercise 90   You can use this exercise to review the following concepts.

• Recognizing a geometric figure generated by the terminal points of three vectors
• Vector addition
• Scalar multiplication

The figure given in the solution is a sample figure and is provided on a transparency. However, the end result, the terminal points of the vectors are collinear, is the same. As you go over the solution, you can review vector operations such as addition and scalar multiplication.

Solution

The terminal points of the vectors $tv, u + tv$ and $su + tv$ are collinear.

Section 11.3   The Dot Product of Two Vectors

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Exercises 43–50 (43–50 in Calculus 8/e)
These exercises used to be separate blocks. We combined them under the direction line for old 47–50. Vector $w_i$ is no longer given. Also, we reordered the exercises and wrote some of the vectors using standard unit vector notation.

We also revised the example that corresponds to these exercises (Example 4 on page 787). In the example, we changed the vectors involved and redrew Figure 11.30 to show that the initial point of $w_i$ is not $(4, 3)$, but $(0, 0)$. 

Exercises 75 and 76 (New)
To give students more practice doing work problems, we added Exercises 75 and 76.

Exercises 81–84 (79–82 in Calculus 8/e)
We rewrote the direction line to include a new part (a) that asks students to find all points of intersection of the graphs of the two equations.

Capstone
Page 790, Exercise 58 You can use this exercise to review the following concepts.
- Definition of dot product
- Angle between two vectors (Theorem 11.5)
- Definition of orthogonal vectors
- Acute and obtuse angles

Go over the definition of dot product and Theorem 11.5. Then go over the solution below. You can illustrate these results using Figure 11.25 on page 785 and with the following examples.

(a) Example 2(b), page 785
(b) Exercise 15, page 789
(c) Example 2(a), page 785

Solution
(a) Orthogonal, \( \theta = \frac{\pi}{2} \)
(b) Acute, \( 0 < \theta < \frac{\pi}{2} \)
(c) Obtuse, \( \frac{\pi}{2} < \theta < \pi \)

Section 11.4 The Cross Product of Two Vectors in Space
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Exercises 11–16 (11–16 in Calculus 8/e)
We reordered the exercises to improve the grading.

Exercises 17–20 (17–20 in Calculus 8/e)
Some students were confused by the y-axis tick label “4” (some thought it meant that \( \mathbf{u} = (3, 5, 4) \)), so we deleted the 4.

Capstone
Page 799, Exercise 54 You can use this exercise to review the following concepts.
- Definition of cross product
- Geometric properties of the cross product (Theorem 11.8)
Review the definition of cross product (see page 792) and the first geometric property of the cross product (see Theorem 11.8, page 794). Then go over the solution below. If desired, you can do an example. Let \( A(0, 0, 0), B(1, 2, 3), \) and \( C(-3, 0, 0) \) be vertices of a triangle. The vectors for two sides of the triangle are

\[
\langle 1 - 0, 2 - 0, 3 - 0 \rangle = \langle 1, 2, 3 \rangle \text{ and } \langle -3 - 0, 0 - 0, 0 - 0 \rangle = \langle -3, 0, 0 \rangle.
\]

The cross product is

\[
\langle 1, 2, 3 \rangle \times \langle -3, 0, 0 \rangle = \langle 0, -9, -6 \rangle.
\]

So, the vector \( \langle 0, -9, -6 \rangle \) is perpendicular to triangle \( ABC \).

**Solution**

Form the vectors for two sides of the triangle, and compute their cross product.

\[
\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \times \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle
\]

### Section 11.5 Lines and Planes in Space

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**Exercises 3 and 4 (New)**

To give students more practice with lines and points in space, we added Exercises 3 and 4.

**Exercises 27–30 (25 and 26 in Calculus 8/e)**

To give students more practice, we added Exercises 28 and 30.

**Exercises 39 and 40 (New)**

To give students more practice with planes and points in space, we added Exercises 39 and 40.

**Exercises 83–86 (75 and 76 in Calculus 8/e)**

To give students more practice, we added Exercises 83 and 84.

**Exercises 91 and 92 (81 and 82 in Calculus 8/e)**

We rewrote the direction line to include a new part (a) that asks students to find the angle between the two planes.

**Capstone**

**Page 810, Exercise 118** You can use this exercise to review the following concepts.

- Parametric equations of a line (Theorem 11.11)
- Symmetric equations of a line
- Standard equation of a plane in space (Theorem 11.12)
- General form of an equation of a plane in space

Go over the definitions of each concept listed above (see pages 800 and 801). Then note which equation or set of equations matches the correct choice.

**Solution**

(a) Matches (iii)

(b) Matches (i)

(c) Matches (iv)

(d) Matches (ii)
Section 11.6 Surfaces in Space

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Exercises 19–32 (19–30 in Calculus 8/e)
To give students more practice, we added Exercises 22 and 23.

Capstone

Page 821, Exercise 58 You can use this exercise to review the following concepts.

- Recognizing the generating curve for a cylinder
- Recognizing equations for cylindrical surfaces

Go over the definition of a cylinder and equations of cylinders on page 812. Then go over the solution. A graph of the cylinder is shown below and is provided on a transparency. The generating curve is a parabola in the \(xz\)-plane. The rulings of the cylinder are parallel to the \(y\)-axis.

Solution

In the \(xz\)-plane, \(z = x^2\) is a parabola.

In the three-space, \(z = x^2\) is a cylinder.

11.7 Cylindrical and Spherical Coordinates

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Exercises 7–12 (7–12 in Calculus 8/e)
We reordered the exercises to improve the grading.

Exercises 21–28 (21–28 in Calculus 8/e)
We reordered the exercises to improve the grading.

Capstone

Page 828, Exercise 98 You can use this exercise to review the following concepts.

- Cylindrical coordinates and surfaces in space
- Spherical coordinates and surfaces in space

Go over the cylindrical coordinate system (see page 822). As you go over part (a), refer to Figures 11.69 and 11.70. Then go over the spherical coordinate system (see page 825). As you go over part (b), refer to Figure 11.76.
Solution

(a) $r = a$ Cylinder with $z$-axis symmetry
   $\theta = b$ Plane perpendicular to $xy$-plane
   $z = c$ Plane parallel to $xy$-plane

(b) $\rho = a$ Sphere
   $\theta = b$ Vertical half-plane
   $\phi = c$ Half-cone
Chapter 11 Project

Bond Angles

The study of the properties of molecules involves determining a molecule’s three-dimensional, geometrical arrangement or molecular structure. Molecular structure is described in terms of bond distances and bond angles. A bond distance is defined as the distance of the straight line connecting the nuclei of two bonded atoms. A bond angle is the angle between any two bonded distances that include a common atom.

Exercises

Use the figure at the right as a representation of a methane molecule, \( CH_4 \). The hydrogen atoms are located at \((0, 0, 0), (1, 1, 0), (1, 0, 1), \) and \((0, 1, 1)\), and the carbon atom is located at \( \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \).

1. Find the component form of \( u \).

2. Find \( \|u\| \).

3. Find the component form of \( v \).

4. Find \( \|v\| \).

5. Find \( u \cdot v \).

6. Use your answers from Exercises 1–5 to determine the bond angle between the hydrogen atoms located at \((1, 1, 0)\) and \((0, 1, 1)\).

7. Determine the bond distance between the hydrogen atoms located at \((1, 1, 0)\) and \((0, 1, 1)\).

8. Find \( \|w\| \) and \( \|z\| \).

In Exercises 9–13, determine the bond angle and the distance between the hydrogen atoms located at the given points.

9. \((0, 0, 0)\) and \((1, 1, 0)\)

10. \((1, 0, 1)\) and \((1, 1, 0)\)

11. \((0, 1, 1)\) and \((0, 0, 0)\)

12. \((0, 1, 1)\) and \((1, 0, 1)\)

13. \((0, 0, 0)\) and \((1, 0, 1)\)

14. Classify the triangles formed by connecting the nuclei of two hydrogen atoms and the carbon atom of a methane molecule. Explain.