Chapter 1  Limits and Their Properties

Chapter Summary

Section Topics

1.1  A Preview of Calculus—Understand what calculus is and how it compares with precalculus. Understand that the tangent line problem is basic to calculus. Understand that the area problem is also basic to calculus.

1.2  Finding Limits Graphically and Numerically—Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

1.3  Evaluating Limits Analytically—Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using dividing out and rationalizing techniques. Evaluate a limit using the Squeeze Theorem.

1.4  Continuity and One-Sided Limits—Determine continuity at a point and continuity on an open interval. Determine one-sided limits and continuity on a closed interval. Use properties of continuity. Understand and use the Intermediate Value Theorem.

1.5  Infinite Limits—Determine infinite limits from the left and from the right. Find and sketch the vertical asymptotes of the graph of a function.

Chapter Comments

Section 1.1 gives a preview of calculus. On pages 43 and 44 of the textbook are examples of some of the concepts from precalculus extended to ideas that require the use of calculus. Review these ideas with your students to give them a feel for where the course is heading.

The idea of a limit is central to calculus. So you should take the time to discuss the tangent line problem and/or the area problem in this section. Exercise 11 of Section 1.1 is yet another example of how limits will be used in calculus. A review of the formula for the distance between two points can be found in Appendix C.

The discussion of limits is difficult for most students the first time that they see it. For this reason you should carefully go over the examples and the informal definition of a limit presented in Section 1.2. Stress to your students that a limit exists only if the answer is a real number. Otherwise, the limit fails to exist, as shown in Examples 3, 4, and 5 of this section. You might want to work Exercise 67 of Section 1.2 with your students in preparation for the definition of the number $e$ coming up in Section 5.1. You may choose to omit the formal definition of a limit.

Carefully go over the properties of limits found in Section 1.3 to ensure that your students are comfortable with the idea of a limit and also with the notation used for limits. By the time you get to Theorem 1.6, it should be obvious to your students that all of these properties amount to direct substitution.

When direct substitution for the limit of a quotient yields the indeterminate form $\frac{0}{0}$, tell your students that they must rewrite the fraction using legitimate algebra. Then, do at least one problem using dividing out techniques and another using rationalizing techniques. Exercises 60 and 62 of Section 1.3 are examples of other algebraic techniques needed for the limit problems. You need to go over the Squeeze Theorem, Theorem 1.8, with your students so that you can use it to prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$. The proof of Theorem 1.8 and many other theorems can be found in Appendix A.

Your students need to memorize both of the results in Theorem 1.9 as they will need these facts to do problems throughout the textbook. Most of your students will need help with Exercises 65–76 in this section.
Continuity, Section 1.4, is another idea that often puzzles students. However, if you describe a continuous function as one in which you can draw the entire graph without lifting your pencil, the idea seems to stay with them. Distinguishing between removable and non-removable discontinuities will help students to determine vertical asymptotes.

To discuss infinite limits, Section 1.5, remind your students of the graph of the function\( f(x) = \frac{1}{x} \) studied in Section P.3. Be sure to make your students write a vertical asymptote as an equation, not just a number. For example, for the function \( y = \frac{1}{x} \), the vertical asymptote is \( x = 0 \).

### Section 1.1 A Preview of Calculus

#### Tips and Tools for Problem Solving

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Although we made changes to the section exercises, no changes were made based on the data.

**Capstone**

Page 47, Exercise 10  You can use this exercise to review the following concepts.

- Understanding what calculus is
- Instantaneous rate of change

Instantaneous rate of change will be discussed in later chapters (e.g., Chapters 2 and 3) and in numerous exercises.

While the answers to this question may vary, a sample answer is given in the solution below. You may want to ask students to define instantaneous rate of change. (We discussed average rate of change in Section P.2. You may want to contrast instantaneous rate of change with average rate of change.) Ask students if instantaneous rate of change is found with or without calculus. (See chart on page 43 of the text.)

**Solution**

Answers will vary. *Sample answer:* The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

### Section 1.2 Finding Limits Graphically and Numerically

#### Tips and Tools for Problem Solving

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Although we made changes to the section exercises, no changes were made based on the data.

**Capstone**

Page 57, Exercise 64  You can use this exercise to review the following concepts.

- The value of \( f \) at \( c \) has no bearing on the existence of the limit of \( f(x) \) as \( x \) approaches \( c \).
- The value of the limit of \( f(x) \) as \( x \) approaches \( c \) has no bearing on the value of \( f \) at \( c \).

Students need to know these concepts because the idea of a limit is central to calculus.
In addition to the solution given below, you could ask students for examples. For part (a), you could ask students to create functions such that \( f(2) = 4 \) where \( \lim_{x \to 2} f(x) \) exists and \( \lim_{x \to 2} f(x) \) does not exist. For part (b), you could ask students to create functions such that \( \lim_{x \to 2} f(x) = 4 \) where \( f(2) = 4 \) and \( f(2) \neq 4 \). Finally, take the opportunity to emphasize the following point (see page 49 of the text): the existence or nonexistence of \( f(x) \) at \( x = c \) has no bearing on the existence of the limit of \( f(x) \) as \( x \) approaches \( c \).

**Solution**

(a) No. The fact that \( f(2) = 4 \) has no bearing on the existence of the limit of \( f(x) \) as \( x \) approaches 2.

(b) No. The fact that \( \lim_{x \to 2} f(x) = 4 \) has no bearing on the value of \( f \) at 2.

**Section 1.3 Evaluating Limits Analytically**

**Tips and Tools for Problem Solving**

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We noticed in the data that students are having trouble with limits involving trigonometric functions (see note on Exercises 65–76, 81–84) and rationalizing numerators (see Exercises 55–58, 77, 78). If you do not have time to review these concepts, encourage students to study the following material in the appendix or in *Precalculus*, 7th edition, by Larson and Hostetler.

- Trigonometric functions: Appendix C.3
- Rationalizing numerators: *Precalculus* Appendix A.2

**Exercises 5–22 (5–22 in Calculus 8/e)**

We revised Exercises 19–22 and reordered the block of exercises to improve the grading.

**Exercises 49–64 (49–62 in Calculus 8/e)**

To give students more practice, we added Exercises 49 and 50. We also revised Exercises 51 and 52 and reordered Exercises 55 and 57 (formerly 53 and 55, respectively) to improve the grading.

**Exercises 65–76, 81–84 (67–82 in Calculus 8/e)**

Most of your students will need help with Exercises 65–76. These exercises require the ability to rewrite trigonometric expressions using algebra, and the proper application of Theorem 1.9. If necessary, have your students study Appendix C.3: Review of Trigonometric Functions.

**Capstone**

Page 69, Exercise 116 You can use this exercise to review the following concept.

- Using a strategy for finding limits

Students need to know this concept because the idea of a limit is central to calculus.

Ask students if the limit of \( f(x) \) as \( x \) approaches 2 can be evaluated by direct substitution. They should understand that this limit *cannot* be evaluated by direct substitution. Ask them how to evaluate this limit. One way is to redefine the function and then use Theorem 1.7. Let \( g(x) = 3 \) for all values of \( x \). Because \( \lim_{x \to 2} g(x) = 3 \), by Theorem 1.7 \( \lim_{x \to 2} f(x) = 3 \). (You can also use the approach suggested in the solution on the next page.) You can reinforce this conclusion using the graph of \( f \) (see the transparency) and by creating a table. Point out to students that the value of \( f \) at \( x = 2 \) is irrelevant. (You may want to do this by asking students if this value is relevant. If you do this, ask them before finding the limit. This reviews a concept taught in Section 1.2.)
Solution

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 3 \]

The value of \( f \) at \( x = 2 \) is irrelevant.

Section 1.4  Continuity and One-Sided Limits

Tips and Tools for Problem Solving

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As in Section 1.3, we noticed in the data that students are having trouble with limits involving trigonometric functions and rationalizing denominators and numerators. Students also had trouble with limits involving rational functions. Encourage students to sharpen their algebra skills by studying the following material in the appendix or in Precalculus, 7th edition, by Larson and Hostetler.

- Trigonometric functions: Appendix C.3
- Rationalizing numerators: Precalculus Appendix A.2
- Simplifying rational expressions: Precalculus Appendix A.4

Exercises 7–26 (7–24 in Calculus 8/e)
To give students more practice, we added Exercises 7 and 8.

Exercises 35–60 (33–54 in Calculus 8/e)
To give students more practice, we added Exercises 35–37 and 39.

Exercises 63–68 (57–60 in Calculus 8/e)
To give students more practice, we added Exercises 63 and 64.

Capstone

Page 81, Exercise 98  You can use this exercise to review the following concepts.

- Describing the difference between a removable discontinuity and a nonremovable discontinuity
- Writing a function that has a removable discontinuity
- Writing a function that has a nonremovable discontinuity
- Writing a function that has a removable discontinuity and a nonremovable discontinuity

Students will need to know these concepts when they study concepts such as vertical asymptotes (Section 1.5), improper integrals (Section 8.8), and functions of several variables (Section 13.2).

Discuss the difference between a removable discontinuity and a nonremovable discontinuity. Use the solution for part (a) given on the next page to illustrate a function with a nonremovable discontinuity. (The graph of this function and the others in the solution are available as transparencies.) Note that it is not possible to remove the discontinuity by redefining the function. When you illustrate a removable discontinuity in part (b), note that the function can be redefined to remove the discontinuity. Ask students how they would do this. \( \text{Let } f(-4) = 1. \) Finally, illustrate part (c). Show students the function and the graph, and then ask them if there are any removable or nonremovable discontinuities. Once they have properly identified \( x = -4 \) as removable and \( x = 4 \) as nonremovable, ask them to redefine \( f \) to remove the discontinuity at \( x = -4. \) \( \text{Let } f(-4) = 0. \)
Solution

A discontinuity at \( c \) is removable if the function \( f \) can be made continuous at \( c \) by appropriately defining (or redefining) \( f(c) \). Otherwise, the discontinuity is nonremovable.

(a) \( f(x) = \frac{|x - 4|}{x - 4} \)

(b) \( f(x) = \frac{\sin(x + 4)}{x + 4} \)

(c) \( f(x) = \begin{cases} 
0, & x < -4 \\
1, & x = -4 \\
0, & -4 < x < 4 \\
1, & x \geq 4 
\end{cases} \)

\( x = 4 \) is nonremovable. \( x = -4 \) is removable.

Section 1.5 Infinite Limits

Tips and Tools for Problem Solving

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As in Section 1.4, we noticed in the data that students are having trouble with limits involving trigonometric functions, rationalizing denominators and numerators, and limits involving rational functions. Refer to the notes in Section 1.4 for the material students should review.


In Exercises 13–32, we reordered the exercises to improve the grading. In Exercises 37–54, we added Exercises 37 and 38 and revised Exercises 39, 41, 43, and 45. Also, we revised Example 1 on page 84 by adding an analytical approach as well as showing a graphical approach to determining an infinite limit. We feel this approach will give students a better understanding of the concept.

Capstone

Page 89, Exercise 64 You can use this exercise to review the following concept.

- Finding the vertical asymptotes of the graph of a function

Students need to understand this concept because it will be used many times as they study calculus.
In class, you can ask students that believe the graph of \( f \) has a vertical asymptote at \( x = 1 \) to prove it. For those that think this is not always true, ask them to provide a counterexample. A sample answer is given in the solution, and you can use the transparency to illustrate a counterexample. If desired, you could also ask students the following.

(a) At which \( x \)-values (if any) is \( f \) not continuous?

(b) Which of the discontinuities are removable?

**Solution**

No, it is not always true. Consider \( p(x) = x^2 - 1 \). The function

\[
 f(x) = \frac{x^2 - 1}{x - 1} = \frac{p(x)}{x - 1}
\]

has a hole at \((1, 2)\), not a vertical asymptote.
Chapter 1 Project
Medicine in the Bloodstream
A patient's kidneys purify 25% of the blood in her body in 4 hours.

Exercises
In Exercises 1–3, a patient takes one 16-milliliter dose of a medication.
1. Determine the amount of medication left in the patient's body after 4, 8, 12, and 16 hours.
2. Notice that after the first 4-hour period, $\frac{3}{4}$ of the 16 milliliters of medication is left in the body, after the second 4-hour period, $\frac{9}{16}$ of the 16 milliliters of medication is left in the body, and so on. Use this information to write an equation that represents the amount $a$ of medication left in the patient's body after $n$ 4-hour periods.
3. Can you find a value of $n$ for which $a$ equals 0? Explain.

In Exercises 4–9, the patient takes an additional 16-milliliter dose every 4 hours.
4. Determine the amount of medication in the patient's body immediately after taking the second dose.
5. Determine the amount of medication in the patient's body immediately after taking the third and fourth doses. What is happening to the amount of medication in the patient's body over time?
6. The medication is eliminated from the patient's body at a constant rate. Sketch a graph that shows the amount of medication in the patient's body during the first 16 hours. Let $x$ represent the number of hours and $y$ represent the amount of medication in the patient's body in milliliters.
7. Use the graph in Exercise 6 to find the limits.
   (a) $\lim_{x \to 4^-} f(x)$
   (b) $\lim_{x \to 4^+} f(x)$
   (c) $\lim_{x \to 12^-} f(x)$
   (d) $\lim_{x \to 12^+} f(x)$
8. Discuss the continuity of the function represented by the graph in Exercise 6. Interpret any discontinuities in the context of the problem.
9. The amount of medication in the patient's body remains constant when the amount eliminated in 4 hours is equal to the additional dose taken at the end of the 4-hour period. Write and solve an equation to find this amount.